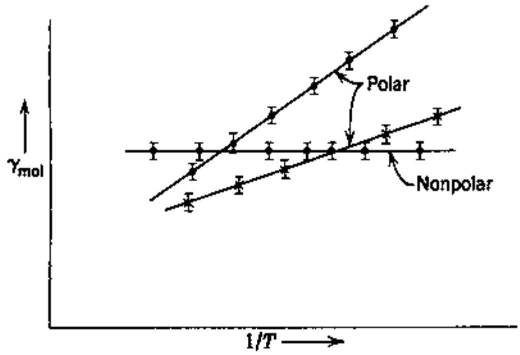

Figure 4.10 Variation of molecular polarizability γ_{mol} with temperature for polar and nonpolar substances: γ_{mol} versus T^{-1}.

This shows a temperature dependence of the form $(a + b/T)$ so that the two types of polarization can be separated experimentally, as indicated in Fig. 4.10. For "polar" molecules, such as HCl and H_2O, the observed permanent dipole moments are of the order of an electronic charge times 10^{-8} cm, in accordance with molecular dimensions.

4.7 Electrostatic Energy in Dielectric Media

In Section 1.11 we discussed the energy of a system of charges in free space. The result obtained there,

$$W = \frac{1}{2} \int \rho(\mathbf{x})\Phi(\mathbf{x})\, d^3x \tag{4.83}$$

for the energy due to a charge density $\rho(\mathbf{x})$ and a potential $\Phi(\mathbf{x})$ cannot in general be taken over as it stands in our macroscopic description of dielectric media. The reason becomes clear when we recall how (4.83) was obtained. We thought of the final configuration of charge as being created by assembling bit by bit the elemental charges, bringing each one in from infinitely far away against the action of the then existing electric field. The total work done was given by (4.83). With dielectric media, work is done not only to bring real (macroscopic) charge into position, but also to produce a certain state of polarization in the medium. If ρ and Φ in (4.83) represent macroscopic variables, it is certainly not evident that (4.83) represents the total work, including that done on the dielectric.

To be general in our description of dielectrics, we will not initially make any assumptions about linearity, uniformity, etc., of the response of a dielectric to an applied field. Rather, let us consider a small change in the energy δW due to some sort of change $\delta\rho$ in the macroscopic charge density ρ existing in all space. The work done to accomplish this change is

$$\delta W = \int \delta\rho(\mathbf{x})\Phi(\mathbf{x})\, d^3x \tag{4.84}$$

where $\Phi(\mathbf{x})$ is the potential due to the charge density $\rho(\mathbf{x})$ already present. Since $\nabla \cdot \mathbf{D} = \rho$, we can relate the change $\delta\rho$ to a change in the displacement of $\delta\mathbf{D}$:

$$\delta\rho = \nabla \cdot (\delta\mathbf{D}) \tag{4.85}$$

Then the energy change δW can be cast into the form:

$$\delta W = \int \mathbf{E} \cdot \delta\mathbf{D}\, d^3x \tag{4.86}$$

where we have used $\mathbf{E} = -\nabla\Phi$ and have assumed that $\rho(\mathbf{x})$ was a localized charge distribution. The total electrostatic energy can now be written down formally, at least, by allowing $\mathbf{D}$ to be brought from an initial value $\mathbf{D} = 0$ to its final value $\mathbf{D}$:

$$W = \int d^3x \int_0^D \mathbf{E} \cdot \delta\mathbf{D} \tag{4.87}$$

If the medium is *linear*, then

$$\mathbf{E} \cdot \delta\mathbf{D} = \tfrac{1}{2}\delta(\mathbf{E} \cdot \mathbf{D}) \tag{4.88}$$

and the total electrostatic energy is

$$W = \frac{1}{2}\int \mathbf{E} \cdot \mathbf{D}\, d^3x \tag{4.89}$$

This last result can be transformed into (4.83) by using $\mathbf{E} = -\nabla\Phi$ and $\nabla \cdot \mathbf{D} = \rho$, or by going back to (4.84) and assuming that ρ and Φ are connected linearly. Thus we see that (4.83) *is valid macroscopically only if the behavior is linear.* Otherwise the energy of a final configuration must be calculated from (4.87) and might conceivably depend on the past history of the system (hysteresis effects).

A problem of considerable interest is the change in energy when a dielectric object with a linear response is placed in an electric field whose sources are fixed. Suppose that initially the electric field $\mathbf{E}_0$ due to a certain distribution of charges $\rho_0(\mathbf{x})$ exists in a medium of electric susceptibility ϵ_0, which may be a function of position (for the moment ϵ_0 is not the susceptibility of the vacuum). The initial electrostatic energy is

$$W_0 = \frac{1}{2}\int \mathbf{E}_0 \cdot \mathbf{D}_0\, d^3x$$

where $\mathbf{D}_0 = \epsilon_0\mathbf{E}_0$. Then with the sources fixed in position a dielectric object of volume V_1 is introduced into the field, changing the field from $\mathbf{E}_0$ to $\mathbf{E}$. The presence of the object can be described by a susceptibility $\epsilon(\mathbf{x})$, which has the value ϵ_1 inside V_1 and ϵ_0 outside V_1. To avoid mathematical difficulties we can imagine $\epsilon(\mathbf{x})$ to be a smoothly varying function of position that falls rapidly but continuously from ϵ_1 to ϵ_0 at the edge of the volume V_1. The energy now has the value

$$W_1 = \frac{1}{2}\int \mathbf{E} \cdot \mathbf{D}\, d^3x$$

where $\mathbf{D} = \epsilon\mathbf{E}$. The difference in the energy can be written:

$$\begin{aligned} W &= \frac{1}{2}\int (\mathbf{E} \cdot \mathbf{D} - \mathbf{E}_0 \cdot \mathbf{D}_0)\, d^3x \\ &= \frac{1}{2}\int (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{D} \cdot \mathbf{E}_0)\, d^3x + \frac{1}{2}\int (\mathbf{E} + \mathbf{E}_0) \cdot (\mathbf{D} - \mathbf{D}_0)\, d^3x \end{aligned} \tag{4.90}$$

The second integral can be shown to vanish by the following argument. Since $\nabla \times (\mathbf{E} + \mathbf{E}_0) = 0$, we can write

$$\mathbf{E} + \mathbf{E}_0 = -\nabla\Phi$$

Then the second integral becomes:

$$I = -\frac{1}{2}\int \nabla\Phi \cdot (\mathbf{D} - \mathbf{D}_0)\, d^3x$$

Integration by parts transforms this into

$$I = \frac{1}{2}\int \Phi\nabla \cdot (\mathbf{D} - \mathbf{D}_0)\, d^3x = 0$$

since $\nabla \cdot (\mathbf{D} - \mathbf{D}_0) = 0$ because the source charge density $\rho_0(\mathbf{x})$ is assumed unaltered by the insertion of the dielectric object. Consequently the energy change is

$$W = \frac{1}{2}\int (\mathbf{E} \cdot \mathbf{D}_0 - \mathbf{D} \cdot \mathbf{E}_0)\, d^3x \tag{4.91}$$

The integration appears to be over all space, but is actually only over the volume V_1 of the object, since, outside V_1, $\mathbf{D} = \epsilon_0\mathbf{E}$. Therefore we can write

$$W = -\frac{1}{2}\int_{V_1} (\epsilon_1 - \epsilon_0)\mathbf{E} \cdot \mathbf{E}_0\, d^3x \tag{4.92}$$

If the medium surrounding the dielectric body is free space, then using the definition of polarization $\mathbf{P}$, (4.92) can then be expressed in the form:

$$W = -\frac{1}{2}\int_{V_1} \mathbf{P} \cdot \mathbf{E}_0\, d^3x \tag{4.93}$$

where $\mathbf{P}$ is the polarization of the dielectric. This shows that the energy density of a dielectric placed in a field $\mathbf{E}_0$ whose sources are fixed is given by

$$w = -\tfrac{1}{2}\mathbf{P} \cdot \mathbf{E}_0 \tag{4.94}$$

This result is analogous to the dipole term in the energy (4.24) of a charge distribution in an external field. The factor $\frac{1}{2}$ is due to the fact that (4.94) represents the energy density of a polarizable dielectric in an external field, rather than a permanent dipole. It is the same factor $\frac{1}{2}$ that appears in (4.88).

Equations (4.92) and (4.93) show that a dielectric body will tend to move toward regions of increasing field $\mathbf{E}_0$ provided $\epsilon_1 > \epsilon_0$. To calculate the force acting we can imagine a small generalized displacement of the body $\delta\xi$. Then there will be a change in the energy δW. Since the charges are held fixed, there is no external source of energy and the change in field energy can be interpreted as a change in the potential energy of the body. This means that there is a force acting on the body:

$$F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_Q \tag{4.95}$$

where the subscript Q has been placed on the partial derivative to indicate that *the sources of the field are kept fixed*.

In practical situations involving the motion of dielectrics the electric fields are often produced by a configuration of electrodes held at *fixed potentials* by connection to an external source such as a battery. To maintain the potentials constant as the distribution of dielectric varies, charge will flow to or from the battery to the electrodes. This means that energy is being supplied from the external source, and it is of interest to compare the energy supplied in that way with the energy change found above *for fixed sources* of the field. We will treat only linear media so that (4.83) is valid. It is sufficient to consider small changes in an existing configuration. From (4.83) it is evident that the change in energy accompanying the changes $\delta\rho(\mathbf{x})$ and $\delta\Phi(\mathbf{x})$ in charge density and potential is

$$\delta W = \frac{1}{2}\int (\rho\ \delta\Phi + \Phi\ \delta\rho)\ d^3x \tag{4.96}$$

Comparison with (4.84) shows that, if the dielectric properties are not changed, the two terms in (4.96) are equal. If, however, the dielectric properties are altered,

$$\epsilon(\mathbf{x}) \to \epsilon(\mathbf{x}) + \delta\epsilon(\mathbf{x}) \tag{4.97}$$

the contributions in (4.96) are not necessarily the same. In fact, we have just calculated the change in energy brought about by introducing a dielectric body into an electric field whose sources were fixed ($\delta\rho = 0$). Equal contributions in (4.96) would imply $\delta W = 0$, but (4.91) or (4.92) are not zero in general. The reason for this difference lies in the existence of the polarization charge. The change in dielectric properties implied by (4.97) can be thought of as a change in the polarization-charge density. If then (4.96) is interpreted as an integral over both free and polarization-charge densities (i.e., a microscopic equation), the two contributions are always equal. However, it is often convenient to deal with macroscopic quantities. Then the equality holds only if the dielectric properties are unchanged.

The process of altering the dielectric properties in some way (by moving the dielectric bodies, by changing their susceptibilities, etc.) in the presence of electrodes at fixed potentials can be viewed as taking place in two steps. In the first step the electrodes are disconnected from the batteries and the charges on them held fixed ($\delta\rho = 0$). With the change (4.97) in dielectric properties, the energy change is

$$\delta W_1 = \frac{1}{2}\int \rho\ \delta\Phi_1\ d^3x \tag{4.98}$$

where $\delta\Phi_1$ is the change in potential produced. This can be shown to yield the result (4.92). In the second step the batteries are connected again to the electrodes to restore their potentials to the original values. There will be a flow of charge $\delta\rho_2$ from the batteries accompanying the change in potential* $\delta\Phi_2 = -\delta\Phi_1$. Therefore the energy change in the second step is

$$\delta W_2 = \frac{1}{2}\int (\rho\ \delta\Phi_2 + \Phi\ \delta\rho_2)\ d^3x = -2\delta W_1 \tag{4.99}$$

*Note that it is necessary merely to know that $\delta\Phi_2 = -\delta\Phi_1$ on the electrodes, since that is the only place where free charge resides.

since the two contributions are equal. In the second step we find the external sources changing the energy in the opposite sense and by twice the amount of the initial step. Consequently the net change is

$$\delta W = -\frac{1}{2}\int \rho\, \delta\Phi_1\, d^3x \tag{4.100}$$

Symbolically

$$\delta W_V = -\delta W_Q \tag{4.101}$$

where the subscript denotes the quantity held fixed. If a dielectric with $\epsilon/\epsilon_0 > 1$ moves into a region of greater field strength, the energy increases instead of decreases. For a generalized displacement $d\xi$ the mechanical force acting is now

$$F_\xi = +\left(\frac{\partial W}{\partial \xi}\right)_V \tag{4.102}$$

References and Suggested Reading

The derivation of the macroscopic equations of electrostatics by averaging over aggregates of atoms is presented in Chapter 6 and by

Rosenfeld, Chapter II
Mason and Weaver, Chapter I, Part III
Van Vleck, Chapter I

Rosenfeld also treats the classical electron theory of dielectrics. Van Vleck's book is devoted to electric and magnetic susceptibilities. Specific works on electric polarization phenomena are those of

Böttcher
Debye
Fröhlich

Boundary-value problems with dielectrics are discussed in all the references on electrostatics in Chapters 2 and 3.

Our treatment of forces and energy with dielectric media is brief. More extensive discussions, including forces on liquid and solid dielectrics, the electric stress tensor, electrostriction, and thermodynamic effects, may be found in

Abraham and Becker, Band 1, Chapter V
Durand, Chapters VI and VII
Landau and Lifshitz, *Electrodynamics of Continuous Media*
Maxwell, Vol. 1, Chapter V
Panofsky and Phillips, Chapter 6
Stratton, Chapter II

Problems

4.1 Calculate the multipole moments q_{lm} of the charge distributions shown as parts a and b. Try to obtain results for the nonvanishing moments valid for all l, but in each case find the first *two* sets of nonvanishing moments at the very least.