CHAPTER 3

CONSERVATION LAWS

In this chapter we will look at the basic "conservation" laws of mechanics, one of which, conservation of momentum, has already been briefly mentioned in Chapter 1. The considerations of that chapter might lead us to consider a "system of particles" consisting of

(a) certain particles $c_1, \dots c_K : \mathbb{R} \to \mathbb{R}^3$,

(b) with positive masses $m_1, \dots, m_K \in \mathbb{R}$,

(c) functions $\mathbf{F}_i^e : \mathbb{R} \to \mathbb{R}^3$,

(d) functions $\mathbf{F}_{ij} = -\mathbf{F}_{ji} : \mathbb{R} \to \mathbb{R}^3$,

with the following basic property: If we set

$$\mathbf{F}_i = \mathbf{F}_i^e + \sum_j \mathbf{F}_{ij},$$

then

$$\mathbf{F}_i = m_i \cdot c_i{}''.$$

Here $\mathbf{F}_i^e(t)$ represents an "external force" on the particle c_i at time t, while the $\mathbf{F}_{ij}(t)$ represent "internal forces" between $c_i(t)$ and $c_j(t)$, satisfying Newton's third law, and consequently $\mathbf{F}_i(t)$ represents the total force on the particle c_i at time t. For forces satisfying the "strong version" of Newton's third law (page 25), condition (d) should also stipulate that $\mathbf{F}_{ij}(t)$ is a multiple of $c_i(t) - c_j(t)$.

Of course, in practice, the $\mathbf{F}_i^e$ and $\mathbf{F}_{ij}$ will often have simple expressions in terms of other functions. For example, the external force $\mathbf{F}_i^e(t)$ might be of the form $m_i \cdot \mathbf{f}(c_i(t))$ for a vector field $\mathbf{f}$ on $\mathbb{R}^3$, e.g., the gravitational attraction due to some external body, while the internal forces $\mathbf{F}_{ij}$ might be some function of m_i, m_j and the distance $|c_i - c_j|$.

Our more general definition allows all sorts of more complicated situations, for example one where the force $\mathbf{F}_i^e$ depends not only on t and $c_i(t)$ but on the whole collection of $\{c_j(t)\}$. A simple instance would be a system of many particles representing a space ship with rocket propulsion, where the direction of the force depends on the particular angle at which the space ship is rotated at any particular time. (Presumably, the space ship has more than one rocket, so that it can steer).

Conservation of momentum. Although conservation of momentum, as stated in Chapter 1, involved only internal forces, we can easily state a generalization allowing for external forces. We set $\mathbf{F} = \sum_i \mathbf{F}_i^e$, the total external force.

1. PROPOSITION (MOMENTUM LAW). The derivative of the total momentum is the total external force,

$$\mathbf{F} = \left(\sum_i m_i \cdot \mathbf{v}_i\right)'.$$

Here we have to regard the various $\mathbf{F}_i^e$ simply as elements of $\mathbb{R}^3$, rather than as tangent vectors at different points of $\mathbb{R}^3$, and similarly for the $\mathbf{v}_i$.

This formulation gains considerable significance when we introduce the concept of the **center of mass** of the system $\{c_i\}$, which represents the "average" position of the particles c_i weighted according to their masses:

$$C = \frac{\sum_i m_i \cdot c_i}{\sum_i m_i}.$$

More precisely, we should define the center of mass as the particle consisting of the path C with the mass $M = \sum_i m_i$.

If all $\mathbf{F}_i^e = 0$, so that $\sum_i m_i \cdot \mathbf{v}_i$ is constant, then $C'' = \frac{1}{M}\sum_i m_i \cdot c_i'' = \frac{1}{M}\sum_i m_i \cdot \mathbf{v}_i' = \frac{1}{M}\left(\sum_i m_i \cdot \mathbf{v}_i\right)'$, so that we also have $C'' = 0$. Thus, C' is constant; in other words, the center of mass moves with uniform velocity.

More generally, we have

2. PROPOSITION. If $\mathbf{F} = \sum_i \mathbf{F}_i^e$ is the total external force, then

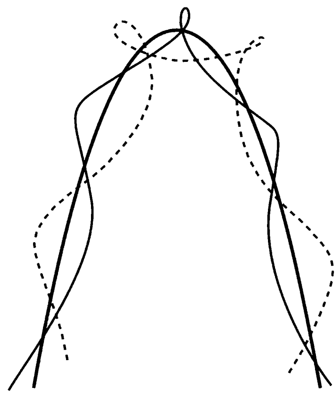
$$\mathbf{F} = M \cdot C'',$$

so that the center of mass particle simply moves as if it were acted upon by the total force $\mathbf{F}$.

PROOF. We have

$$\begin{aligned} M \cdot C'' &= \sum_i m_i \cdot c_i'' \\ &= \sum_i \mathbf{F}_i \\ &= \sum_i \mathbf{F}_i^e + \sum_i \sum_j \mathbf{F}_{ij} \\ &= \sum_i \mathbf{F}_i^e. \quad ❖ \end{aligned}$$

Although the "particle" C might not be one of the particles in our system, this result is seldom regarded as particularly "theoretical"—instead it allows us to get a very simple picture of very complex phenomena. For example, in the figure on page 10, showing a rod executing a complicated revolving motion, the center of mass, which does happen to be a point on the rod in this case, simply moves in a parabola, just like a point mass. A striking illustration may be obtained with a time-exposure photograph taken when a baton is tossed in the air, with lights at the ends and the center of mass, giving a picture like this:

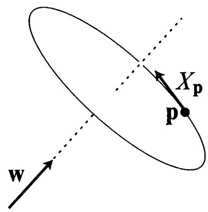


We usually think of a rod as a "rigid body", a concept whose analysis we have still shied away from. At first, that might seem to make the result even more impressive: in a real rod, with all sorts of complicated intermolecular forces, which make it approximately "rigid", but not truly so, it is still true that the center of mass moves according to a simple law. But that is a somewhat misleading way of construing the result, since rigidity of the rod is required in order to identify its center of mass with a particular point of the rod, on which we can attach one of the lights.

Center of mass is often called "center of gravity", a concept that goes back at least to Archimedes (cf. the Prologue). These concepts are not identical except in a uniform gravitational field—which applies, of course, to reasonable sized objects on the earth's surface—but the difference is often ignored.

Conservation of angular momentum. The use of the cross-product $\times$ at the beginning of the previous chapter could be regarded simply as a convenient abbreviation for manipulations with determinants. But there is a more important reason why this special product of $\mathbb{R}^3$ is significant.

For any vector $\mathbf{w} \in \mathbb{R}^3$, consider the one-parameter family of maps $B(t)\colon \mathbb{R}^3 \to \mathbb{R}^3$, where $B(t)$ is a counter-clockwise rotation through an angle of $t|\mathbf{w}|$ radians around the axis through $\mathbf{w}$ [choosing an orientation $(\mathbf{v}_1, \mathbf{v}_2)$ of the plane perpendicular to $\mathbf{w}$ so that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w})$ is the usual orientation of $\mathbb{R}^3$]. Now consider the vector field generated by this one-parameter family. In other words, for each $\mathbf{p} \in \mathbb{R}^3$ consider the curve $B_{\mathbf{p}}(t) = B(t)(\mathbf{p})$, and then look at the tangent vector $X_{\mathbf{p}}$ of this curve at 0.



To compute $X_{\mathbf{p}}$ geometrically, we note that $X_{\mathbf{p}}$ is clearly perpendicular to both $\mathbf{p}$ and $\mathbf{w}$. Its length is also easy to determine. When $\mathbf{p}$ happens to lie in the plane perpendicular to $\mathbf{w}$, as in (a), the point $\mathbf{p}$ rotates in a circle of radius $|\mathbf{p}|$, and $X_{\mathbf{p}}$ has length $|\mathbf{p}| \cdot |\mathbf{w}|$. More generally (b), the point $\mathbf{p}$ rotates in a circle of radius $|\mathbf{p}| \cdot |\mathbf{w}| \cdot \sin\theta$, where θ is the angle between $\mathbf{w}$ and $\mathbf{p}$. Thus, $X_{\mathbf{p}}$ is just the geometrically defined cross-product $\mathbf{w} \times \mathbf{p}$.

(a) (b)

For an analytic determination of $X_{\mathbf{p}} = B_{\mathbf{p}}'(0)$, we note that since the $B_{\mathbf{p}}(t)$ are all orthogonal, and $B_{\mathbf{p}}(0) = I$, the derivative $B_{\mathbf{p}}'(0)$ is skew-adjoint, with a skew-symmetric matrix M, which we will write in the form

$$M = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Then the vector $X_{\mathbf{p}}$ is the 3-tuple whose transpose $X_{\mathbf{p}}{}^{\mathbf{t}}$ is given by

$$\begin{aligned} X_{\mathbf{p}}{}^{\mathbf{t}} &= M \cdot (p_1, p_2, p_3)^{\mathbf{t}} \\ &= \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \cdot (p_1, p_2, p_3)^{\mathbf{t}} \\ &= (-p_2\omega_3 + p_3\omega_2, -p_3\omega_1 + p_1\omega_3, -p_1\omega_2 + p_2\omega_1)^{\mathbf{t}}. \end{aligned}$$

Setting $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, we then have $X_{\mathbf{p}} = \boldsymbol{\omega} \times \mathbf{p}$. Moreover, $\boldsymbol{\omega}$ is easy to identify, because

$$B_{\mathbf{w}}(t) = \mathbf{w} \text{ for all } t \quad \Longrightarrow \quad 0 = X_{\mathbf{w}} = \boldsymbol{\omega} \times \mathbf{w},$$

so $\boldsymbol{\omega}$ must be a multiple of $\mathbf{w}$, and it easy to check, by considering some specially chosen vector, that in fact $\boldsymbol{\omega} = \mathbf{w}$. Thus, one might say that the cross-product $\times$ is special to $\mathbb{R}^3$ because $n = 3$ is the only dimension where $\mathrm{O}(n)$ has dimension n. More to the point, we have

PROPOSITION. The vector fields in $\mathbb{R}^3$ generated by rotations about an axis are of the form $\mathbf{p} \mapsto \boldsymbol{\omega} \times \mathbf{p}$ for $\boldsymbol{\omega} \in \mathbb{R}^3$.

For a particle c with velocity vector $\mathbf{v}$ we can consider the function $c \times \mathbf{v}$ from $\mathbb{R}$ to $\mathbb{R}^3$, which is called the **angular velocity** of the particle; if $c(t) = (x(t), y(t), z(t))$ for functions x, y, and z, then the angular velocity of c is

$$(\mathrm{A}_c) \qquad (yz' - y'z,\ x'z - xz',\ xy' - x'y).$$

For a particle whose mass is m, the cross-product $\mathbf{L} = c \times m\mathbf{v}$ is called its **angular momentum**. The angular velocity and momentum just defined are "with respect to the origin 0": for any other point P, the angular momentum with respect to P is the cross-product

$$\mathbf{L}_P = (c - P) \times m\mathbf{v}.$$

For a system of particles $(c_1, \dots, c_K)$ we define the angular momentum $\mathbf{L}$ of the system, with respect to 0, as

$$\mathbf{L} = \sum_{i=1}^{K} c_i \times m_i \mathbf{v}_i;$$

here it is naturally necessary to consider all $c_i \times m_i \mathbf{v}_i$ as vectors at a single point, rather than as tangent vectors at different points. Note that the equation $\mathbf{L}' = \sum_{i=1}^{K} (c_i \times m_i {c_i}')'$ reduces to

$$(\mathrm{L}') \qquad \mathbf{L}' = \sum_{i=1}^{K} c_i \times m_i {c_i}''.$$

More generally, we define the angular momentum $\mathbf{L}_P$ with respect to P as $\mathbf{L}_P = \sum_{i=1}^{K}(c_i - P) \times m_i\mathbf{v}_i$. In particular, suppose we take P to be the center of mass C of the system (this means that we may be considering the angular momentum with respect to different points at different times). Letting $M = \sum_i m_i$, the "mass" of the particle C, we then have

$$\begin{aligned}\sum_i m_i c_i \times \mathbf{v}_i &= \sum_i m_i(c_i - C) \times \mathbf{v}_i \;+\; \sum_i m_i C \times \mathbf{v}_i \\ &= \mathbf{L}_C + \left[C \times \left(\textstyle\sum_i m_i\mathbf{v}_i\right)\right] \\ &= \mathbf{L}_C + [C \times MC'],\end{aligned}$$

so that we can write

$$\mathbf{L} = \mathbf{L}_C + (C \times MC').$$

The vector $\mathbf{L}_C$, the angular momentum with respect to the center of mass, is also called the "rotational angular momentum", so our equation says that the total angular momentum $\mathbf{L}$ is the sum of the rotational angular momentum $\mathbf{L}_C$ and the angular momentum of the center of mass with respect to 0.

If instead of a momentum vector we consider an arbitrary force $\mathbf{F}$ at a point c, the cross-product

$$\boldsymbol{\tau} = c \times \mathbf{F}$$

is called the **torque** of the force with respect to 0, while $\boldsymbol{\tau}_P = (c - P) \times \mathbf{F}$ is the torque with respect to P. (Although I have used the physicists' $\mathbf{L}$ for angular momentum, I couldn't bring myself to use the standard $\mathbf{N}$ for torque.)

Similarly, we define the torque of a system of forces on a system of particles; here it is again necessary to consider the individual torques as being vectors at one point, even though we naturally think of the forces as being applied at different points.

3. PROPOSITION (ANGULAR MOMENTUM LAW). If our system satisfies the strong form of the third law, then the total torque is the derivative of the total angular momentum,

$$\boldsymbol{\tau} = \mathbf{L}'.$$

PROOF. We have

$$\begin{aligned}\mathbf{L}' &= \sum_i c_i \times m_i c_i'' \qquad \text{by equation } (\mathrm{L}') \\ &= \sum_i c_i \times \mathbf{F}_i \\ &= \sum_i c_i \times \mathbf{F}_i^e + \sum_i \sum_j c_i \times \mathbf{F}_{ij}\end{aligned}$$

$$= \boldsymbol{\tau} + \sum_i \sum_j c_i \times \mathbf{F}_{ij}.$$

The strong form of the third law allows us to write

$$\mathbf{F}_{ij} = \lambda_{ij}(c_i - c_j),$$

with $\lambda_{ij} = \lambda_{ji}$, so we have

$$\sum_i \sum_j c_i \times \mathbf{F}_{ij} = \sum_i \sum_j \lambda_{ij}[c_i \times c_i \ - \ c_i \times c_j],$$

which vanishes, since $c_i \times c_i = 0$, while $c_i \times c_j = -c_j \times c_i$ and $\lambda_{ij} = \lambda_{ji}$. ❖

Easy manipulations give us the more general

4. COROLLARY. For any point P,

$$\boldsymbol{\tau}_P = \mathbf{L}_P{}'$$

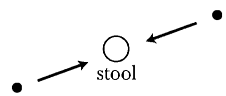
In particular, of course, if the torque is 0, then angular momentum is conserved. This certainly happens in the special case of a single particle moving under a *central force*, where the external force $\mathbf{F}$ is a multiple of c, so that $\boldsymbol{\tau} = c \times \mathbf{F} = 0$. This was noted by Newton as the first Corollary of his Proposition 1 (page 55):

> Corollary 1. In nonresisting spaces, the velocity of a body attracted to an immobile center is inversely as the perpendicular dropped from that center to the straight line which is tangent to the orbit.

As we saw in Chapter 2, the particle actually stays in a plane, and if we have $c(t) = (x(t), y(t), 0)$, say, then conservation of angular momentum just says, by equation (A_c), that $xy' - yx'$ is constant. Even the somewhat more general rule that angular momentum is conserved in the absence of external forces was not stated until quite some time afterwards, and this law was known for a long time simply as "the law of areas", or *Flächensatz* in German.

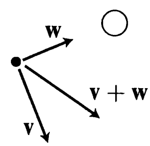
The evocative term "torque" (from the Latin *torquere*, to twist) was not introduced until the 19th century. Before that, the cross-product $c \times \mathbf{F}$ was called the **moment** of the force $\mathbf{F}$ at the point c, with respect to 0. Here "moment" is being used in the sense of "importance" or "significance" (e.g., a matter of great moment), this significance having been noted long before in terms of the law of the lever. Correspondingly, angular momentum was known as the "moment of momentum", a term which has not yet been totally expunged.

A standard elementary illustration of the law of conservation of angular momentum is provided by a person seated on a rotating stool with arms extended out holding weights, and then increasing the speed of the spin, often quite dra-



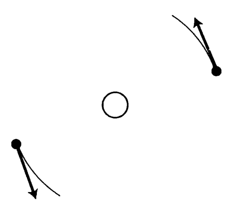
matically, simply by pulling the weights inward. Similarly, ice-skaters speed up their turns by pulling their arms in; divers, starting their dive with a small angular momentum, do rapid somersaults by pulling their arms and knees in; and gymnasts do all sorts of tricks.

By the way, without appealing to conservation of angular momentum we can explain the speed-up as a simple consequence of the parallelogram rule for forces, or even for velocities: the sum of the velocity **v** that the weight already has and the velocity **w** that it acquires as a result of the inward pull is the



diagonal of the rectangle spanned by these two, and consequently has a greater length.

In these examples, we merely altered the given non-zero angular momentum, but something interesting occurs even when we start with angular momentum 0. Moving the weights along a circle in one direction contributes a certain amount of angular momentum to the system of weights-plus-person, which must be



countered by an opposite amount of angular momentum in the system, so the seated person must rotate in the opposite direction. At the end of the motion, when the weights are no longer being rotated, the person will have stopped rotating, but will be facing in a different direction; cats use this mechanism to land on their paws even when dropped from an upside-down position.

In this respect, rotation is quite different from linear motion. A system cannot change its position using only internal forces, and no external forces. On a perfectly frictionless ice surface you can change the direction in which you are facing, but you can't move the position of your center of mass. (Of course, you can forcefully exhale, providing yourself with rocket propulsion, making use of the fact that the air inside your lungs is a part of your system that you aren't attached to—or you could just throw your coat away.)

The momentum law and the angular momentum law are the first two great conservation laws of mechanics, and they apply to all mechanical systems, although their application may also require further understanding about rigid bodies and other matters. We should also mention that they are vector equations, so, for example, if the x-component of the total external force $\mathbf{F}$ is 0, then the x-component of the total momentum is constant; or to put it more generally, if the total external force $\mathbf{F}$ is 0 in one particular direction, then the total momentum in that direction is also constant. This probably doesn't seem particularly useful, but the analogue for angular momentum definitely can be (cf. Problem 5): if the the torque is some direction is 0, then the component of the angular momentum in that direction is constant.

The third conservation law is quite different: it is both much more special and much more general.

Conservation of energy: kinetic and potential energy. As Galileo had noted, for a body falling under the acceleration of gravity, the distance s that it travels after being released from rest satisfies $s = at^2$ for some a, so that $v = s' = 2at$ satisfies $v^2 = \text{constant} \cdot s$, expressing v in terms of the distance traveled, rather than in terms of the time traveled.

More generally, suppose a body falls due to a force that depends only on its height x from the earth's surface,

$$mx''(t) = -f(x(t)),$$

for some function f; as in Problem 2-6, we add the $-$ sign so that a positive f corresponds to an attractive force *towards* the earth's surface, the direction in which x *decreases*. Although we may not be able to solve for x explicitly, we can still get out information about $v = x'$. We use the obvious trick of multiplying both sides of the above equation by $x'(t)$, so that the right side becomes a derivative,

$$mx'(t)x''(t) = -f(x(t)) \cdot x'(t) = -(F \circ x)'(t) \qquad \text{for } F' = f,$$

and then observe that the left side is also a derivative, so that we get

$$\left(\tfrac{1}{2}mx'^2\right)' = -(F \circ x)'.$$

The quantity $T = \frac{1}{2}mv^2$ is called the **kinetic energy** of the body, so if we let $x_i = x(t_i)$ for $i = 0, 1$ and $v_i = v(t_i) = x'(t_i)$, we have

$$(*) \qquad T(t_1) - T(t_0) = \tfrac{1}{2}mv_1{}^2 - \tfrac{1}{2}mv_0{}^2 = -F(x_1) + F(x_0),$$

so that the difference $T(t_1) - T(t_0)$ of the kinetic energy at two times depends only on the heights at the two times.

As an aside, we point out that an alternative approach is suggested if we know that we are looking for an expression for v in terms of x. As on page 75, we use Leibnizian notation to transform $mv'(t) = -f(x(t))$ into a formula for dv/dx:

$$\frac{dv}{dx} = \frac{dv}{dt} \Big/ \frac{dx}{dt} = \frac{-f(x)}{mv}$$

$$-f(x) = mv\frac{dv}{dx} = \frac{1}{2}m\frac{dv^2}{dx}$$

$$\tfrac{1}{2}mv^2 = \int f(x)\,dx.$$

We've been discussing a one-dimensional situation, or equivalently one in which our force always points in one direction, but the same conclusion holds for the more general case of a radially symmetric central force. We introduce polar coordinates (r, θ) for the plane in which the motion takes place, and let $\mathbf{r}$ be the unit vector field pointing toward the origin, while $\boldsymbol{\theta}$ is the perpendicular unit vector field. At any point x our radially symmetric central force has the value $-f(|x|)\mathbf{r}$ for some function f. If for convenience we introduce the usual "abuse of notation" of allowing r to stand for $r \circ c$ and θ to stand for $\theta \circ c$, then

$$\begin{aligned}
\tfrac{1}{2}m(v^2)' &= \tfrac{1}{2}m \cdot \langle \mathbf{v}, \mathbf{v}\rangle' = m\langle \mathbf{v}, \mathbf{v}'\rangle = \langle \mathbf{v}, m\mathbf{v}'\rangle \\
&= \langle \mathbf{v}, -(f \circ r)\mathbf{r}\rangle \\
&= \langle r'\mathbf{r} + \theta'\boldsymbol{\theta}, -(f \circ r)\mathbf{r}\rangle \\
&= -(f \circ r)r' = -(F \circ r)' \qquad \text{for } F' = f.
\end{aligned}$$

We thus have

$$(**) \qquad T(t_1) - T(t_0) = -F(r(t_1)) + F(r(t_0)),$$

so that the difference in kinetic energy at two times depends only on the distances from the origin at the two times.

Nowadays this result is usually stated rather differently. If we let $V : \mathbb{R}^3 \to \mathbb{R}$ be the function

$$V = F \circ r,$$

then $(**)$ becomes

$$\begin{aligned} T(t_1) - T(t_0) &= -F(r(c(t_1))) + F(r(c(t_0))) \\ &= -V(c(t_1)) + V(c(t_0)), \end{aligned}$$

and if we choose a fixed t_0 we find that

$$(***) \qquad T(t) + V(c(t)) \;\text{ is constant.}$$

The quantity $V(p)$ is called the **potential energy** of the particle at $p \in \mathbb{R}^3$, so equation $(***)$, called **conservation of energy**, states that for radially symmetric central forces the sum of the kinetic energy and the potential energy of a particle is constant throughout its path. The important point here is that $V(c(t))$ depends only on the position $c(t)$ of the particle, not on the path c itself.

Obviously, the function V is only determined up to a constant. For elementary problems involving free falling bodies near the surface of the earth, it is customary to consider V to be 0 on the earth's surface, so that its value when the body is released from some height is positive. As the body falls, its potential energy decreases as its kinetic energy increases. This accords with the usual interpretation of V as the kinetic energy that the body "potentially" has, i.e., the kinetic energy that it can acquire by being released, and allowed to fall to earth.

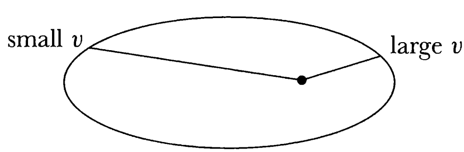
In the case of an inverse square force, a body falling radially toward the center, with distance $r(t)$ from the center given by

$$r''(t) = -K/r(t)^2,$$

has

$$v(t) = r'(t) = K/r \implies V(p) = -\frac{K}{r(p)} + \text{constant.}$$

It is convenient to take the constant to be 0, so that V is 0 at ∞. For a planet moving in an ellipse, V is larger (though negative) at points further from the sun, so the kinetic energy is smaller there (as implied by Kepler's second law).



More generally, we define a force $\mathbf{F} = (F_1, F_2, F_3)$ to be **conservative**, with **potential energy** function V, if the conservation of energy equation

$$\text{(C)} \qquad \tfrac{1}{2}m\langle \mathbf{v}(t), \mathbf{v}(t)\rangle + V(c(t)) = \text{constant}$$

holds for all particles $c(t)$ moving under the force $\mathbf{F}$. For the standard coordinate system (x^1, x^2, x^3) on $\mathbb{R}^3$, differentiating gives

$$\begin{aligned} 0 &= \langle \mathbf{v}(t), m\mathbf{v}'(t)\rangle + \frac{d}{dt}V(c(t)) \\ &= \langle \mathbf{v}(t), \mathbf{F}(c(t))\rangle + \sum_{i=1}^{3} \frac{\partial V}{\partial x^i}(c(t)) \cdot c_i{}'(t). \end{aligned}$$

Choosing a path c with $c(0) = p$, and evaluating at $t = 0$, we obtain

$$0 = \langle \mathbf{v}(0), \mathbf{F}(p)\rangle + \sum_{i=1}^{3} \frac{\partial V}{\partial x^i}(p) \cdot c_i{}'(0).$$

Since, under the standard identification of tangent vectors of $\mathbb{R}^3$ with $\mathbb{R}^3$ itself, we also have

$$\mathbf{v}(0) = (c_1{}'(0), c_2{}'(0), c_3{}'(0)),$$

we see, by choosing c with only one $c_i{}'(0) \neq 0$, that we must have

$$\text{(C}'\text{)} \qquad \mathbf{F} = (F_1, F_2, F_3) = -\left(\frac{\partial V}{\partial x^1}, \frac{\partial V}{\partial x^2}, \frac{\partial V}{\partial x^3}\right),$$

which physicists usually write as

$$\mathbf{F} = -\operatorname{grad} V.$$

Equivalently,

$$\left\langle \mathbf{F}, \frac{\partial}{\partial x^i}\right\rangle = F_i = -\frac{\partial}{\partial x^i}(V),$$

and thus, more generally, for any tangent vector $\mathbf{v}$ we have

$$\langle \mathbf{F}, \mathbf{v}\rangle = -\mathbf{v}(V),$$

for the usual operation of a tangent vector $\mathbf{v}$ on a function.

Conversely, suppose that $\mathbf{F}$ satisfies (C′). For any field $\mathbf{F} = (F_1, F_2, F_3)$ we have

$$\begin{aligned} T(t_1) - T(t_0) &= \tfrac{1}{2}m{v_1}^2 - \tfrac{1}{2}m{v_0}^2 \\ &= \int_{t_0}^{t_1} \langle \mathbf{v}(t), \mathbf{F}(c(t))\rangle \, dt \\ &= \int_{t_0}^{t_1} \sum_{i=1}^{3} c_i{}'(t) \cdot F_i(c(t)), \end{aligned}$$

and if we introduce the 1-form

$$\omega = F_1\, dx^1 + F_2\, dx^2 + F_3\, dx^3$$

and let γ be the curve $\gamma = c|[t_0, t_1]$, this can be written as

$$(*) \qquad T(t_1) - T(t_0) = \int_\gamma \omega.$$

Now if $\mathbf{F}$ satisfies (C′), then $\omega = -dV$, so we have

$$\begin{aligned} T(t_1) - T(t_0) &= \int_\gamma -dV \\ &= -V(c(t_1)) + V(c(t_0)), \end{aligned}$$

which implies that $\mathbf{F}$ is conservative, with potential function V.

The quantity

$$\int_{t_0}^{t_1} \langle \mathbf{v}(t), \mathbf{F}(c(t))\rangle \, dt = \int_\gamma \omega$$

is called the **work** done by the force $\mathbf{F}$ on the particle c as it moves along the path γ. We have just seen that for conservative forces this depends only on the end-points of the path. As a simple example, consider a closed elliptical path, on a time interval $[0, T_0]$, of a particle moving under an inverse square force $\mathbf{F}$. The total work done by $\mathbf{F}$ along this path must be 0, since that is the total work done on the interval $[0, 0]$, which has the same end-points.

In general, of course, there usually won't be more than one trajectory between two points. The more interesting situation—and the one that connects with our every-day notion of work— arises when we consider the work done by a force as we move it along some other path, i.e., the work that has to be done *against* the force field $\mathbf{F}$ in order to move a particle from one point to another. For example, raising a particle of mass m from height h_0 to height h_1 near the earth's surface requires a total work of $mg(h_1 - h_0)$, no matter what path we take, provided that periods during which the particle is moving downward instead of upwards are regarded as contributing negative work.

This property of conservative forces is usually regarded as the more important one in physics books, so they define a force $\mathbf{F}$ to be conservative if $\int_\gamma \omega$ depends only on the end-points of γ. This immediately implies our previous definition, since we can define

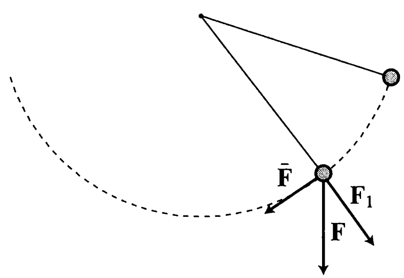
$$V(p) = \int_\gamma \omega$$

where γ is any path from a fixed point p_0 to p, and equation $(*)$ on page 91 immediately leads to the conservation of energy equation[1]

$$\text{(C)} \qquad \tfrac{1}{2}\langle \mathbf{v}(t), \mathbf{v}(t)\rangle + V(c(t)) = \text{constant}.$$

Looking at the calculation for equation $(**)$ on page 88 we see that it still holds if we replace $\mathbf{F}$ by $\mathbf{F} + \mathbf{F}_1$ where $\mathbf{F}_1$ is always perpendicular to $\mathbf{v}$; as physicists would express it, the work done by the extra force $\mathbf{F}_1$ is 0, by the very hypothesis that it is always perpendicular to $\mathbf{v}$.

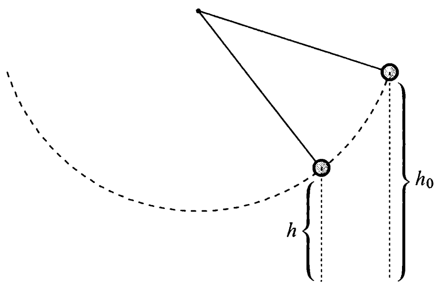
In particular, instead of a body moving under the gravitational force of the earth, consider one suspended by a thread, so that we have a pendulum bob, which we have analyzed (somewhat informally) in Problem 1-17. The total force



on the pendulum bob is then $\bar{\mathbf{F}} = \mathbf{F} - \mathbf{F}_1$, where $\mathbf{F}$ is the conservative force from the gravitation field of the earth, while $\mathbf{F}_1$ is the force that was introduced in our analysis, pointing along the thread. Since $\mathbf{F}_1$ points along the thread, $\langle \mathbf{v}, \mathbf{F}_1\rangle = 0$, so we still have the conservation of energy (C).

[1] For some strange reason, physics books (and even mathematics books) usually eschew this simple direct argument, instead noting that the dependence of $\int_\gamma \omega$ only on the end-points of γ implies that $\int_\gamma \omega = 0$ for all closed γ, so that for all 2-chains σ we have, by Stokes' theorem, $0 = \int_{\partial\sigma} \omega = \int_\sigma d\omega$, and thus we must have $d\omega = 0$. This implies (with proper conditions on the region where $\mathbf{F}$ is defined) that $\omega = -dV$ for some V.

This has the interesting consequence that although we cannot explicitly solve the pendulum equation derived on page 47, we can still say what the speed v



of the bob is at any height h, because we have

$$mgh + \tfrac{1}{2}mv^2 = \text{constant},$$

so we just have to know the height h_0 at which we released the bob, with $v = 0$ (compare Problem 1-17). The pendulum can be regarded as a mechanism that is continually interchanging potential energy and kinetic energy. At the top of the swing the kinetic energy is 0, while at the bottom of the swing, the difference in potential energy has been converted to kinetic energy, just sufficient to raise it up to the same height at which it started.

Similarly, on page 30 we mentioned the usual elementary analysis of a block sliding down an inclined plane, where we assume that the block is acted upon by the force of gravity $\mathbf{F}$ and another force $-\mathbf{F}_1$ perpendicular to the inclined plane. Thus this argument works in that case also, and the kinetic energy $\frac{1}{2}mv^2$ at the bottom must again be gh. So the speed of the block when it reaches the bottom must be the same as if it fell straight down, which agrees with our calculations, since the block's acceleration along the inclined plane is only $\sin\alpha$ of its falling acceleration, but it has $1/\sin\alpha$ as far to go. In the same way, instead of a pendulum bob hanging from a string, we could just as well allow the bob to slide along a plane with a circular profile, or indeed any profile, if we could really provide a frictionless surface.

Aside from its obvious physical interest, conservation of energy is important mathematically as a "first integral" of the laws of motion, i.e., an equation involving only first derivatives, rather than the second derivatives that appear in Newton's laws—all harking back to our original trick on page 87. In the next chapter, we will derive the result of Problem 2-6 in a more systematic way, starting from conservation of energy, with the sign of the total energy E of an orbit turning out to have a simple geometric significance.

Conservation of energy in collisions. While the role of kinetic energy with respect to conservative forces seems fairly straightforward, there was initially considerable confusion about kinetic energy because of the completely different role that it plays in that simplest, yet most essential, physical phenomenon, the collision of two bodies.

Consider two particles, c_1 and c_2, with masses m_1 and m_2, moving along a straight line with velocities v_1 and v_2; as usual, since our motion is confined to a straight line, we can represent the velocities simply by numbers. It seems natural to ask the question: if they collide, what are their new velocities w_1 and w_2 after the collision?

Conservation of momentum gives us only one equation,

$$(1) \qquad m_1w_1 + m_2w_2 = m_1v_1 + m_2v_2,$$

for the two unknowns w_1 and w_2, so it obviously can't determine an answer to the question, even under special circumstances, like the case where $m_1 = m_2$ and $v_2 = 0$, so that we have a moving object colliding with a stationary one of the same mass. One possible solution would be $w_1 = 0$ and $w_2 = v_1$, so that the first body stops and imparts all its motion to the second (something close to this happens when two steel balls collide). On the other hand, the second body might be "soft", like clay, so that it yields on impact, losing its shape and adhering to the first body, with the two then moving together as one (alternatively, we might consider bodies that will stick after contact because of glue, as on page 26, or perhaps carts with couplings that cause them to move as one after an impact), and in this case the final velocity of the two bodies will simply be $v_1/2$, just another possible solution of the infinitely many.

Elementary physics textbooks need to provide problems that have answers, of course, so, in the manner of a host nonchalantly introducing a celebrity at a party, they will often unobtrusively insert a new definition: a collision is called "completely elastic", if we also have conservation of kinetic energy,

$$(2) \qquad \begin{cases} m_1{w_1}^2 + m_2{w_2}^2 = m_1{v_1}^2 + m_2{v_2}^2 \\ \qquad\qquad\text{or} \\ m_1({w_1}^2 - {v_1}^2) = m_2({v_2}^2 - {w_2}^2) \end{cases}$$

Consorting with this new definition we have a contrasting one: a collision is "completely inelastic" if $w_1 = w_2$ (the two bodies stick together).

Once we've made a definition, it's possible to pose all sorts of simple problems about collisions that are "completely elastic", whatever that might mean. In general, writing (1) in the from

$$m_1(v_1 - w_1) = m_2(w_2 - v_2),$$

and dividing into [the second form of] (2), which is permissible so long as we don't have $w_1 = v_1$ (and $w_2 = v_2$), we get

$$w_1 - w_2 = -(v_1 - v_2), \tag{3}$$

and solving (1) and (3) for the unknowns w_1 and w_2 gives

$$(*) \qquad \begin{aligned} w_1 &= \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2 \\ w_2 &= \frac{2m_1}{m_1 + m_2} v_1 - \frac{m_1 - m_2}{m_1 + m_2} v_2; \end{aligned}$$

the other solution, $w_1 = v_1$ and $w_2 = v_2$, is discarded on physical grounds, since it represents the particles moving through each other.

This rather unenlightening formula will appear much more natural when we express it in terms of "center of mass coordinates" (Problem 10). In any event, we obtain the usual obvious cases: if $m_1 = m_2$ and $v_1 = -v_2$, so that we have two particles of equal mass approaching each other with opposite velocities, we get $w_1 = v_2$ and $w_2 = v_1$, so the two particles rebound with the same speeds at which they collided; if $m_1 = m_2$ and $v_2 = 0$, then $w_1 = 0$ and $w_2 = v_1$, so the first particle comes to a stop, while the second proceeds with the velocity of the first.

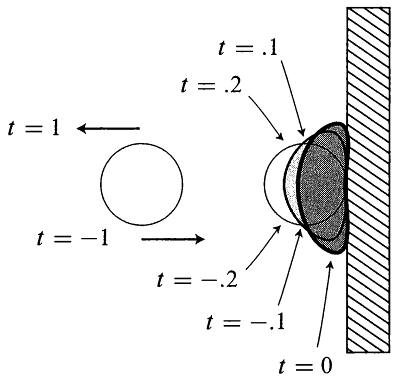
For general collisions, a *coefficient of restitution* e is sometimes defined by

$$(w_1 - w_2) = -e(v_1 - v_2) \qquad \text{or} \qquad e = \frac{w_2 - w_1}{v_1 - v_2},$$

which experimentally seems to be (somewhat) independent of the initial velocities v_1 and v_2. This is usually applied only when the two objects are moving towards each other and rebound in opposite directions after the collision, so that if we have, for example, $v_1 > 0$ and $v_2 < 0$, then also $w_1 < 0$ and $w_2 > 0$, which means that $e \geq 0$.

But having a definition of the coefficient of restitution hardly tell us anything; it simply give us a way of specifying how far experimental results differ from the theoretical ones that we obtain from our *ad hoc* definition of completely elastic collisions. We would like to understand why the modern definition of a completely elastic collision amounts to an idealization of the concept that lurks in the back of our minds when we think of an "elastic" body as one that pops back into shape after being squashed in a collision.

To simplify things, let's start by considering collisions of one body with a wall, whose mass may be regarded as so large that we don't have to worry about its motion. First we take some nice modeling clay, form it into a ball, and hurl it at the wall, where it sticks in some deformed shape. This is clearly an example of a "completely inelastic" collision. Then we throw a rubber ball at the wall. The rubber ball is also squashed when it hits the wall, but, unlike the clay, the



compressed rubber ball restores itself to its old shape, and bounces back in the reverse direction. Of course, it never bounces back with quite the same speed, but the term "completely elastic" was meant to describe an idealization of this situation, where the ball ends up pushing itself back with the same amount of force that caused the compression in the first place, so that it bounces back with the same speed. In this case, of course, we do have conservation of kinetic energy.

The general case of a "completely elastic" collision of two bodies, with velocities v_1 and v_2 along a straight line can be treated in a similar way. In this case, both bodies are deformed, and this deformation will continue until the two bodies have the same velocity u, which, by conservation of momentum, must be

$$u = \frac{m_1 v_2 + m_2 v_2}{m_1 + m_2}. \tag{1}$$

During the compression, the first body's velocity will change from v_1 to u, so the compression will involve decreasing the velocity by the amount $u - v_1$. Consequently, when it decompresses, its velocity is then *increased* from u by the amount $u - v_1$, and of course the same reasoning applies to the second body. So the final velocities w_1 and w_2 are given by

$$\begin{aligned} w_1 &= u + (u - v_1) = 2u - v_1 \\ w_2 &= 2u - v_2. \end{aligned} \tag{2}$$

Using the value of u from (1), we easily find that $m_1w_1{}^2 + m_2w_2{}^2 = m_1v_1{}^2 + m_2v_2{}^2$; in fact, substituting (1) into (2) gives exactly the equations (∗) on page 95 that we obtained by assuming conservation of kinetic energy.

This entire discussion has been limited to "head-on" collisions, but Problems 12 and 13 have some information about the more general case.

Conservation of energy in general. Although "collisions" between atomic particles may be completely elastic, this is virtually never the case for everyday collisions between objects, where we can only hope to come fairly close to complete elasticity with objects like steel or ivory balls, and this is but one example where conservation of kinetic energy fails in general. A completely different example is illustrated by a rocket. Suppose that it is initially at rest, so that the initial kinetic energy is 0 (we assume that the rocket is in space away from any gravitational fields, so that there is no external force on the rocket, and thus no potential energy to consider). Once the rocket has expelled some fuel, so that it and the fuel are both moving, in opposite directions, the kinetic energy clearly isn't 0, since the kinetic energies are non-negative numbers, and therefore can't cancel out like momentum.

A similar phenomenon occurs when one stationary person shoots another with a rifle—the resulting motion of the bullet upsets conservation of kinetic energy (as well as the person being shot at). Since momentum is always conserved, the shooter experiences the recoil of the rifle, which is the negative of the momentum that the bullet obtains, and which will be transferred to the target, hopefully outfitted with a "bullet-proof" vest—the violent effects obtained without such protection are only indirectly a measure of the bullet's momentum, depending more on the fact that it is delivered to such a small area.

Of course, nowadays we would say that the loss of kinetic energy involved in collisions is due to its dissipation as heat, that the increase in kinetic energy of the rocket is due to the conversion of chemical energy in the fuel, and that the increase in kinetic energy of the bullet is similarly due to the conversion of chemical energy in the gunpowder. In all cases, the total energy—when we add up the heat energy and the chemical energy, and all the other types of energy which go into modern physics—is supposed to remain constant.

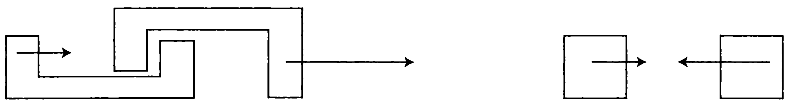
To quote from Feynman [1], the law of *conservation of energy*

> ... states that there is a certain quantity, which we call energy, that does not change in the manifold changes which nature undergoes. ... It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same.

Feynman goes on to discuss this by means of an analogy which is both very instructive and very entertaining, but much too long to quote here, so you should go read, or re-read, it yourself. In fact, chapters 4 through 13 of Feynman [1] may be regarded as a continuing exposition of the role that the concept of energy plays in physics.

In summary, as far as mechanics is concerned, conservation of energy—kinetic plus potential—is an important principal for conservative forces, which are generally the ones we wish to consider. On the other hand, for more complicated phenomena, which involve other forms of "energy", there are no such conservation laws; or, to put it another way, the conservation of energy involves factors which are basically outside the purview of mechanics itself.

In this regard, we may consider once again "completely inelastic" collisions. In addition to the case of a ball of clay hurled at a wall, or even at a more mobile object like a steel cube, we can also consider a completely inelastic collision between two rigid bodies that stick together because of couplings, or perhaps



glue on opposing surfaces. It is easy (Problem 15) to compute the energy lost in the collision; the result was first obtained by General Lazarus Carnot, the father of Sadi Carnot of thermodynamics fame, and has been dubbed the "Carnot energy loss" in Sommerfeld [2], mentioned on page 35.

This energy loss presumably shows up in shock waves coursing through the (not really rigid) bodies, dissipating as heat and sound waves, and possibly in some sort of chemical reaction involving the glue. In a way, this is the exact opposite of the rocket, where a large kinetic energy evolves from none at all—in that case, totally because of a chemical reaction.

A Carnot energy loss that we might judge to be fairly large produces only a small change in temperature, which might be difficult to observe experimentally. For a simple calculation, consider two iron cubes each with a mass of 1 kilogram, which smash together after moving toward each other, each with speeds of 1 meter/second. The energy loss would then be

$$1\,\mathrm{Kg}\,\frac{\mathrm{m}^2}{\mathrm{s}^2} = 1\mathrm{J},$$

by definition of the Joule. The specific heat of iron is

$$.45\ \frac{\mathrm{J}}{\mathrm{g}\,^\circ\mathrm{C}},$$

where °C denotes degrees centigrade, so 1 Joule raises the temperature of 1 gram of iron by .45°C, and our two iron cubes, of total mass 2 kilograms, would have their temperature raised by .45/2000 degrees centigrade. If instead of a speed of 1 m/s, which is only 3.6 km/hr, we chose speeds 30 times as fast, namely 108 km/hr, or roughly 67 miles/hr, then the temperature increase would be 900 times as great, or roughly .2 degrees centigrade; of course, this result holds just as well for two iron cubes of arbitrary masses that end up moving as one.

In mechanics problems we naturally do not expect to determine exactly how such energy losses occur in order to understand the underlying mechanical principles. But this circumstance provides a convenient rug under which all sorts of mysterious energy losses can be swept; a classic example is discussed in the following Addendum.

ADDENDUM 3A

WHIPS AND CHAINS
(Why Easy Physics is So Hard: II)

The progenitor of all those horrible "variable mass" problems introduced in Addendum 1A, which have been used to torment generations of physics students ever since, was a paper by a mathematician, Cayley [1], that begins: "There are a class of dynamical problems which, so far as I am aware, have not been considered in a general manner. The problems referred to ... are those in which the system is continually taking into connexion with itself particles of infinitesimal mass ... For instance, a problem of the sort arises when a portion of a heavy chain hangs over the edge of a table, the remainder of the chain being coiled or heaped up close to the edge of the table ... " (presumably an idealized case of a fine chain with very small links, as in Problem 1-13; notice that in Problem 1-13 the entire chain is always being pulled, so Cayley's problem is quite different).

We can apply the equations $(**)$ and $(**')$ on page 35, where our variable mass is the part of the chain hanging over the table, with the additional links added as the chain falls; since these additional links are initially at rest, the velocity at which they are added, relative to the falling chain, is $-\mathbf{v}$, so this is again a case where $\mathbf{v} + \mathbf{q} = 0$, and our equation is simply

$$\mathbf{F}(t) = (m\mathbf{v})'(t).$$

Taking the uniform density of the chain to be 1 for convenience, if $x(t)$ is the length of chain hanging over the table at time t, then $m(t) = x(t)$, while $\mathbf{F}(t)$ has magnitude $gx(t)$. Thus our equation becomes simply

$$gx = (xx')'.$$

If we set $y = xx'$, so that

$$gx = y' = \frac{dy}{dt},$$

and then use the good old-fashioned trick of writing

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt},$$

we obtain

$$gx = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot x',$$

so that

$$gx^2 = \frac{dy}{dx} \cdot xx' = \frac{dy}{dx} \cdot y,$$

or

$$gx^2\,dx = y\,dy.$$

Thus we have

$$g\frac{x^3}{3} + A = \frac{y^2}{2},$$

and if we assume the initial condition $x'(0) = 0$, we get $A = -ga^3/3$, where a is the initial length hanging over the table. This leads to

$$(x')^2 = \frac{2g}{3}\left(x - \frac{a^3}{x^2}\right). \tag{1}$$

Cayley considered only the special case $a = 0$ (which means that the whole chain is initially on the table, so that it shouldn't fall at all, but presumably he assumed that the result would be a good approximation to the case where a is small). Then (1) becomes $x' = \sqrt{2g/3}\,\sqrt{x}$, and integrating $x'/\sqrt{x} = \sqrt{2g/3}$ yields

$$2\sqrt{x(t)} = \sqrt{2g/3} \cdot t, \quad \text{or} \quad x(t) = (1/6)gt^2.$$

Consequently, the chain is always falling with $1/3$ the acceleration of a freely falling chain; of course, that applies only until the chain has cleared the table, after which its acceleration must simply be g.

At first sight this seems rather weird and unlikely, and we might suspect that it is an artifact of the strange choice $a = 0$. Cayley may have taken this case because he didn't want to deal with the general equation (1), which would require an elliptic integral. But, without solving explicitly, we can still compute from (1) that

$$2x'x'' = \frac{2g}{3}\left(x' + \frac{2a^3x'}{x^3}\right),$$

so that

$$x'' = \frac{g}{3}\left(1 + \frac{2a^3}{x^3}\right).$$

At the beginning ($x = a$) this has the value g, and it then decreases, approaching $g/3$ for long chains, with the same sort of discontinuity as before. (The equation for x'' follows from the previous equation when $x' \neq 0$, (i.e., $x \neq a$), and then for $x = a$ by continuity, although technically we must appeal to an elementary calculus theorem.[1])

[1] If f is continuous at a and $\lim_{x\to a} f'(x)$ exists, then $f'(a)$ exists and $= \lim_{x\to a} f'(x)$. See, e.g., Spivak [1], Theorem 11-7.

To make sense of this perplexing answer, which of course is only approximate for the case of an actual chain, it helps to note that each time another link is added to the falling chain, that link is suddenly yanked from velocity 0 to the velocity of the falling chain, and the resultant increase in momentum must be balanced by a decrease in momentum of the falling chain. As the falling chain gets longer and more massive, one might expect the effect to be less noticeable, but the longer falling chain also has a much greater velocity, so the momentum added to the next link also increases greatly.

This problem has appeared in many standard mechanics books—usually with Cayley's solution, though Sommerfeld [2] hints at the more general solution—not so much for its own sake, but in order to examine the question of conservation of energy.

When a piece of chain of length x is hanging over the table, the potential energy has decreased by $\int_a^x gu\,du = \frac{1}{2}g(x^2 - a^2)$, while the kinetic energy has increased by $\frac{1}{2}x(x')^2$, so that the total change of energy is

$$\Delta E = \tfrac{1}{2}x(x')^2 - \tfrac{1}{2}g(x^2 - a^2).$$

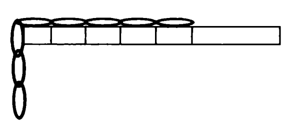
By (1) this is

$$-\tfrac{1}{6}gx^2 + \tfrac{1}{2}ga^2 - \tfrac{1}{3}g\frac{a^3}{x}$$

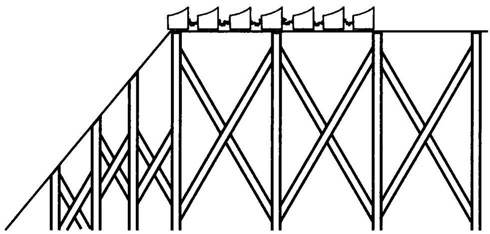
so for all $x \geq a$ it is negative, and increases rapidly as x gets large. It is hardly surprising that conservation of energy does not hold for this solution, since, as Problem 23 shows, it *does* hold for the solution to Problem 1-13.

This loss of energy is explained in terms of the Carnot energy losses each time the dangling chain pulls another link off the table, the point being that this is a completely inelastic collision, since the resulting velocities of the two bodies are the same; the energy loss presumably ends up heating the chain.

It actually turns out to be rather difficult to conduct experiments to check our answer, because one can't get a real chain "coiled or heaped up close to the edge of the table" in such a way that each link is right at the table edge, ready to be added to the falling part; in practice, there is an unpredictable jumble as individual links are released. One way to simulate the conditions of the problem might

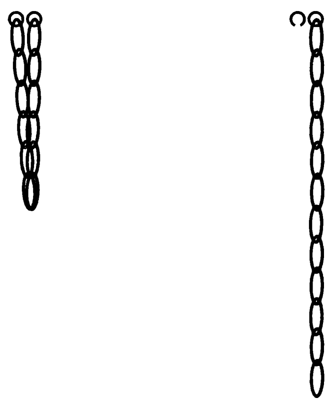


be to lay the additional links along a table made of slats, removing each leftmost slat once the link lying on it has been pulled off. An inexact, but instructive approximation could be provided by a carnival ride, the "whip", where individual cars are lined up close to the edge of a slide, but attached with bunched-up 50′ chains. After the first car is allowed to start sliding, the experience of its



riders is quite different from that of riders in the last car!)

There is another classical problem for which experiments can more easily be carried out. Consider a folded chain, initially hanging by supports at both



ends, and then released at one end. After the chain has finished falling, its total energy (all in the form of potential energy) is much smaller.

The classical description of this situation is that each of the links on the free end of the chain just falls with acceleration g until being jerked to a stop, with the loss of energy being accounted for by the corresponding Carnot energy losses. But a simple experiment shows that the acceleration must be considerably greater than g. It involves only a moderately heavy chain, a single link from such a chain, and two pieces of window glass. The specific chain used in one experiment was 5 feet long ($\approx$ 152 cm), with 50 links, and weighed about 14.3 oz ($\approx$ 405 gm). The thickness of the glass was 3/16 inches ($\approx$ .5 cm).

Opposite ends of one piece of glass were placed on rests of the same height (two copies of a book), and the single link was repeatedly dropped onto it from a height greater than 5 feet, with no apparent ill effect (sometimes the link was initially held horizontally, sometimes vertically). The piece of glass was replaced

with the second, fresh, piece, and the 5 foot long chain was secured so that it hung with only the last link touching the glass. The free end was then raised to the same height as the secured end (a short distance away from it horizontally, so that the chain wouldn't become entangled in itself as it fell) and released. The result was a dramatic shattering of the glass plate.

We can analyze the fall of the chain in this problem in the same way as the Cayley problem, taking as our body with variable mass the falling part of the chain, which is "losing" links to the fixed part. Since these links become stationary as they join the fixed part of the chain, we again have $\mathbf{q} = -\mathbf{v}$, so we still have the case where $\mathbf{v} + \mathbf{q} = 0$, leading to the same equation

$$\mathbf{F}(t) = (m\mathbf{v})'(t).$$

As before, we assume that the chain has uniform density 1. It will also be convenient to assume that the fully extended chain is hanging so that it just touches the ground, and then let x be the height of the free end of the chain, initially having the value x_0 (thus, $x_0 = L$ for a folded chain of length L with both ends initially at the same height). The falling part of the chain has length $x/2$, so our equation becomes

$$g\frac{x}{2} = -\frac{1}{2}(xx')',$$

or simply $gx = -(xx')'$. Setting $y = xx'$ we now have

$$\begin{aligned} gx &= -\frac{dy}{dt} \\ &= -\frac{dy}{dx} \cdot x', \end{aligned}$$

so that

$$gx^2 = -\frac{dy}{dx} \cdot xx' = -\frac{dy}{dx} \cdot y,$$

and hence

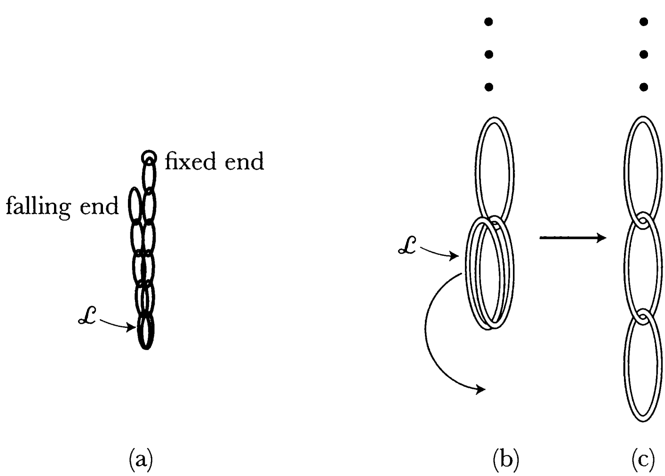
$$g\frac{x^3}{3} + A = -\frac{y^2}{2}.$$

At $t = 0$ we have $x = x_0$ and $x' = 0$, so we get $A = -gx_0{}^3/3$, leading to

$$\text{(A)} \qquad \begin{cases} (x')^2 = \dfrac{2}{3}g\left(\dfrac{x_0{}^3}{x^2} - x\right) \\[2ex] x'' = -\dfrac{g}{3}\left(\dfrac{2x_0{}^3}{x^3} + 1\right), \end{cases}$$

the second equation following by differentiation of the first, as before. Thus the downward acceleration starts at g and then increases, so the released end of the chain falls faster than a freely falling chain.

This increase in acceleration can be explained by considering a link $\mathcal{L}$ of the chain that has just reached the bottom, as in (a). This link has acquired a large velocity, but is now going to be stopped dead in its descent by the part of the



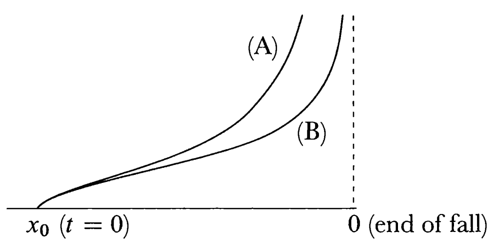
chain on the fixed end, and all that momentum will be used to yank the link around by 180°, as shown magnified in (b) and (c) of the figure. This yanking is going to pull the falling part of the chain even faster. This is basically just the opposite of what happens for the Cayley problem, where the falling chain yanks the next link off the table, resulting in the falling chain having its acceleration reduced; in the current situation the link that was falling, but now becomes part of the fixed chain, yanks the falling chain, resulting in the falling chain having its acceleration increased.

The fact that the falling chain has acceleration greater than g seems to have been first observed by Calkin and March [1]. To explain the results of their experiments (rather more sophisticated than the shattered glass experiment), they completely jettisoned the question of Carnot energy losses, and simply assumed that conservation of energy holds, obtaining the equations (Problem 24)

$$\text{(B)} \qquad \begin{cases} (x')^2 = g\left(\dfrac{{x_0}^2}{x} - x\right) \\[2ex] x'' = -\dfrac{g}{2}\left(\dfrac{{x_0}^2}{x^2} + 1\right), \end{cases}$$

which seem to agree quite well with their experimental results. (In the case of this solution, the increase in velocity is easily explained by the fact that the same amount of energy has to be concentrated in shorter and shorter pieces of chain, so that the velocities must increase.)

In the figure below, comparing the downward speeds for equations (A) and (B), the direction of the x-axis is reversed, so that $x = x_0$, at time $t = 0$, appears on the left, while $x = 0$, at the end of the fall, appears on the right. I do not



know how well the Calkin-March data would match up with equations (A) [for an actual chain, of only finitely many links, either set of equations becomes less reliable near the end of the fall, which is where the solutions diverge the most], or how to choose between them, or whether the solution for a real chain is some sort of compromise between the two. Or, for that matter, how one should treat the same problem when the chain is replaced by a rope.

Note that, with either solution, at the end of the fall the speed and acceleration actually become infinite, or at any rate very large for an actual chain of only finitely many links, where the friction between links also takes its toll. This possibly counter-intuitive behavior is also demonstrated by the crack of a whip;

here the force applied to the whip takes the place of gravity, and the crack of the whip is a shock wave caused by the very large velocity with which the end of the whip is traveling.

A more recent paper examining these questions, Wong and Yasui [1], approves of the Calkin-March solution, dismisses Cayley's solution of his problem, and by extension the one given here, in favor of a conservation of energy solution (Problem 25), and goes on to discuss the folded chain problem in great detail. This paper contains an extensive bibliography of previous solutions to both the folded chain problem and the Cayley problem, which may be very instructive to peruse. But it appears to me that all the conclusions of the paper itself are wrong.

Undoubtedly others will find that all the conclusions in this Addendum are wrong.

> *I must admit to being totally confused. I thought mechanics was a cookbook subject where one uses a few basic principles to translate physics into mathematics, and then revs up the calculus machine and grinds out the answer. I guess your book is intended to cure those of us who have this misapprehension.*
>
> —An eminent mathematician