ADDENDUM 1A

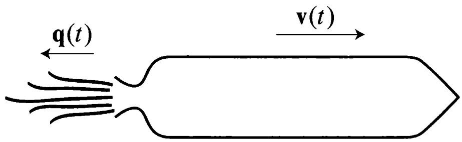
IT ISN'T ROCKET SCIENCE
(Why Easy Physics is So Hard: I)

School children are customarily told that rockets work because of the third law, instead of being provided with the more prosaic explanation that the burning fuel explodes both toward the front of the rocket and toward the rear, with the fuel that explodes toward the front pushing the rocket forward, while the fuel that explodes toward the rear simply escapes out the open end of the rocket.

As a matter of fact, the action of a rocket doesn't follow so clearly from the third law itself as from it's consequence, the conservation of momentum: since the burning fuel has momentum in one direction, the remaining fuel and rocket must have momentum in the other direction in order to conserve the total momentum—namely 0 if the rocket starts at rest. As this example illustrates, one of the main attractions of "conservation" laws is that they often allow us to consider quite complicated situations involving myriad particles at once.

Of course, a few idealizations are required here: rocket fuel is usually a liquid, but it is not unreasonable to regard it as a bunch of particles (which, at the atomic level, it really is); and the (empty) rocket itself is hardly a particle, and we might demand a more careful analysis of how the force exerted on one part gets transmitted to other parts, but we will defer that to a later chapter.

Now it would seem fairly straightforward to get an analytic expression for the motion of a rocket. We'll first consider a rocket in empty space, so that there is no external force acting on it. Let $\mathbf{v}$ be its velocity, and let $m(t)$ be the mass at time t, by which we mean the constant mass of the empty rocket plus the mass of the fuel still in the rocket at time t. Equivalently, $-m' = \mu$ is the rate at which the fuel is burned; this depends ultimately on the chemical characteristics of the fuel, the design of the combustion mechanism, etc., etc. We also need to consider the velocity $\mathbf{q}$ with which the fuel is ejected from the rocket; we'll use $\mathbf{q}$ for the velocity *with respect to the rocket*, so that $\mathbf{q} + \mathbf{v}$ is the velocity with respect to our inertial system. If we ignored $\mathbf{q}$, then we could just as well assume that the fuel was simply being dumped off the rocket ($\mathbf{q} = 0$), which wouldn't result in any motion at all!



The simplest analysis proceeds from the prosaic point of view suggested at the beginning. In a short time interval $[t, t + h]$, the amount of fuel ejected is

$m(t) - m(t+h)$, and therefore the momentum of expelled fuel will be close to $[m(t) - m(t+h)] \cdot \mathbf{q}(t)$. Thus, the momentum of the fuel in the other direction, pushing the rocket forward, will be the negative of this. So the force on the rocket must be the derivative, $m'(t) \cdot \mathbf{q}(t)$, and by the second law this means that

$$(*) \qquad m'(t)\mathbf{q}(t) = m(t)\mathbf{v}'(t).$$

This can also be written as

$$\mathbf{v}'(t) = \frac{m'(t)}{m(t)}\mathbf{q}(t) = \frac{d}{dt}\log(m(t))\mathbf{q}(t).$$

In our example, $\mathbf{v}$ and $\mathbf{q}$ always point along the same straight line, and if we let v and q denote their lengths—the speed of the rocket and the speed of fuel ejection—then, remembering that $\mathbf{q}$ and $\mathbf{v}$ point in opposite directions, we can simply write

$$\text{(R)} \qquad v'(t) = -q(t)\frac{m'(t)}{m(t)} = -q(t)\frac{d}{dt}\log(m(t)).$$

This argument might seem suspect, since we appear to be working in a coordinate system based on the rocket, which is not an inertial system, but that isn't really the case. Although we derived $(*)$ from the "rocket's point of view", at each particular time t_0 we were essentially working in the inertial system that is moving with the same velocity as the rocket at time t_0, and since the derived equation $(*)$ involves only the *change* $\mathbf{v}'(t_0)$ of velocity at time t_0, it holds just as well in the inertial system where we are making our measurements of position. Nevertheless, most physics books avoid tackling an explanation of this sort and instead present the following analysis.

Since the ejection velocity of the fuel with respect to our inertial system is $\mathbf{q}+\mathbf{v}$, in a small time interval $[t, t+h]$, the amount of fuel ejected, $m(t) - m(t+h)$, has a velocity close to $\mathbf{v}(t) + \mathbf{q}(t)$, so the total momentum of this expelled fuel is close to

$$[m(t) - m(t+h)] \cdot \big(\mathbf{v}(t) + \mathbf{q}(t)\big).$$

The derivative at time t of the momentum of the expelled fuel is

$$\lim_{h \to 0} \frac{m(t) - m(t+h)}{h} \cdot \big(\mathbf{v}(t) + \mathbf{q}(t)\big) = -m'(t) \cdot \big(\mathbf{v}(t) + \mathbf{q}(t)\big).$$

Setting this equal to the derivative of $-m(t)\mathbf{v}(t)$, we get

$$\frac{d}{dt}[m(t)\mathbf{v}(t)] = m'(t) \cdot \big(\mathbf{v}(t) + \mathbf{q}(t)\big),$$

which can also be written as our original equation

$$(*) \qquad m'(t)\mathbf{q}(t) = m(t)\mathbf{v}'(t).$$

The funny thing about this problem is that we tend to think of it as a "real-life" problem, involving a continuously changing fuel mass, and then find ourselves in the position of having to use laws that apply only to individual particles. But if we made our "real-life" problem really real, by considering the fuel as a collection of particles being ejected individually in tiny increments of time, then we would view our rocket as receiving tiny changes of velocity at these times, but moving with constant velocity in the intervals between. In other words, our rocket is an inertial system on these intervals, which makes the validity of the first argument much more transparent.

A completely independent source of confusion is offered by some less recent mechanics texts, which like to point out that in special relativity, which we will barely mention in this volume, the mass m of even an individual particle is not constant, but depends upon its velocity; it then turns out that the second law, which we have always stated as

$$\begin{aligned} \mathbf{F} &= (m\mathbf{v})' \\ &= m\mathbf{v}' \end{aligned}$$

has to be corrected to read

$$\mathbf{F} = (m\mathbf{v})' = m'\mathbf{v} + m\mathbf{v}'.$$

Of course, the mass of a particle doesn't vary with velocity in classical mechanics, but when we bring an external force $\mathbf{F}$ into the picture, we might wonder whether this more general equation is the right one to use for a rocket, with variable mass $m(t)$. If $\mathbf{F}$ is an external force on the rocket, e.g., gravity, should we use

$$(1) \qquad \mathbf{F}(t) = m(t)\mathbf{v}'(t)$$

or

$$(2) \qquad \mathbf{F}(t) = (m\mathbf{v})'(t)\,?$$

The short answer is: neither. When $\mathbf{F} = 0$, equation (1) contradicts $(*)$ unless $\mathbf{q} = 0$; this is hardly surprising, since $\mathbf{F}$ is simply the "external" force on the rocket, and we still have to account for the force exerted by the escaping fuel. And (2) likewise contradicts $(*)$ when $\mathbf{F} = 0$ except in the special case where $\mathbf{q} = -\mathbf{v}$.

In fact, using the same reasoning by which we established (∗) we can conclude more generally that when an external force $\mathbf{F}$ acts on the rocket we have

(∗∗)	$\mathbf{F}(t) = m(t)\mathbf{v}'(t) - m'(t)\mathbf{q}(t)$
(∗∗′)	$\mathbf{F}(t) = (m\mathbf{v})'(t) - m'(t)[\mathbf{v}(t) + \mathbf{q}(t)].$

A more complete answer is that the question is completely misleading. It purports to be studying an object, "the rocket", that has a variable mass. But the objects that we really have are the empty rocket, together with a myriad of particles of rocket fuel, some of which are moving along with the empty rocket, while others are moving in the opposite direction.

At any time t the force of the burning fuel acts on the empty rocket together with the part of the fuel still moving along with it. By additivity of mass, this composite object is given the same acceleration as a single object whose mass is what we have called $m(t)$, and which we misleadingly called the "mass of the rocket at time t".

In any case, the formula (∗∗) or (∗∗′) states the proper result and this analysis applies just as well to any "variable mass" problem, where a body's mass is changing as particles are continually dispersed, or are added on, with velocity $\mathbf{q}$ (or velocity $\mathbf{v} + \mathbf{q}$ with respect to an inertial system).

Unfortunately, some textbooks like to claim that the proper law is, in fact, (2), perhaps under the sway of the influential text *Mechanics* by Sommerfeld [2], where Newton's statement of the second law is specifically identified as (2), with the remark that "Newton's formulation ... prophetically turns out to be the correct one."

Naturally, considerable backtracking is required when (2) has been perpetrated as the proper relationship, as shown, for example, by the thorough, and thoroughly confusing, discussion in §I.4 of Sommerfeld's book.

To make matters worse, one class of variable mass problems (∗∗′) reduces to (2)—namely, those in which $\mathbf{v} + \mathbf{q} = 0$. For example, consider a satellite moving in empty space uniformly filed with stationery interplanetary debris that sticks to the satellite when it hits it, thereby increasing the satellite's mass $m(t)$. In this case, (2) is applicable because it amounts to (∗∗′) with $\mathbf{v} + \mathbf{q} = 0$, which merely says that the accumulated debris is initially at rest before it starts moving along with the satellite. (Since our analysis was made in terms of mass removed from a body, it might help to think of the reverse time picture, where bits of the satellite are being expelled with exactly the velocity that they need to become part of the stationery interplanetary debris.)